§ 7.1 rational maps of curves

When a pt is located in the domain of a rational map.

$F: X \dashrightarrow Y$ dominating rational map. $\quad \tilde{F}: k(Y) \hookrightarrow k(X)$, $\quad \mathcal{O}_Q(Y) \subset k(Y)$, $\quad \mathcal{O}_P(X) \subset k(X)$

$F(P) = Q \Rightarrow \tilde{F}(\mathcal{O}_Q(Y)) \subset \mathcal{O}_P(X)$ & $\tilde{F}(m_P(X)) = m_Q(Y)$.

$F(P) = Q \implies$ • $\forall g \in \mathcal{O}_Q(Y) \Rightarrow \tilde{F}(g) = g \circ F$ is defined at P

$\Rightarrow \tilde{F}(g) \in \mathcal{O}_P(X)$

• $\forall g \in m_Q(Y) \Rightarrow \tilde{F}(g)|_P = g(F(P)) = g(Q) = 0$

$\Rightarrow \tilde{F}(g) \in m_P(Y)$.

<u>Def</u>: $(A, m_A), (B, m_B)$ = loc. ring. $A \subset B$. B **dominates** A if $m_A \subseteq m_B$.

Lemma: Let $F: X \dashrightarrow Y$ be a dominating rational map. $\forall P \in X, Q \in Y$.

$$\left.\begin{array}{l} P \in \underbrace{U(F)}_{\text{domain of } F} \\ Q = F(P). \end{array}\right\} \Leftrightarrow \mathcal{O}_P(X) \text{ dominates } \tilde{F}(\mathcal{O}_Q(Y))$$

Pf: $\Rightarrow$): clear

. $\Leftarrow$): $P \in V$, $Q \in W$ affine neighborhood

assume $\Gamma(W) = k[y_1, \dots, y_n]$. $\quad \tilde{F}(y_i) = \frac{a_i}{b_i}$ $\left(\begin{array}{l} a_i, b_i \in \Gamma(V) \\ b_i(P) \neq 0 \end{array} \right)$

$b = b_1 \cdots b_n \Rightarrow \tilde{F}(\Gamma(W)) \subset \Gamma(V_b)$

$\Rightarrow \exists! \ f: V_b \to W$

$\forall g \in \Gamma(W)$, $g(Q) = 0 \Rightarrow g \in m_Q \Rightarrow \tilde{F}(g) \in m_P$

$\Rightarrow g \circ f(P) = \tilde{F}(g)(P) = 0 \Rightarrow f(P) = Q$

Def: C = curve, $p \in C$.

p is called a simple point if $\mathcal{O}_p(C) = $ DVR.

$\mathrm{ord}_p = \mathrm{ord}_p^C : k(C) \to \mathbb{Z}$

the order function on $k(C)$ defined by $\mathcal{O}_p(C)$

C is called nonsingular if every point on C is simple.

- See §3.1 for simple pt on a plane curve
- Thm 1 of §3.2. the definition here agree with that in §3.1 in plane curve case

Def: K/k = field extension. A subring $A \subseteq K$ containing k is called a local ring of K if A is local and $K = \mathrm{Frac}(A)$.

a discrete valuation ring of K is a DVR that is a local ring of K

e.g. V = variety. $P \in V$ Then

1) $\mathcal{O}_P(V)$ is a local ring of $k(V)$

2) V = curve & P simple $\Rightarrow \mathcal{O}_P(V) = $ DVR of $k(V)$.

Thm 1. C = proj. curve. $K = k(C)$. L/K = field ext. R = DVR of L. Assume $R \not\supseteq K$. Then $\exists!\ P \in C$ s.t.

R dominates $\mathcal{O}_P(C)$.

Pf: uniqueness: Suppose R dominates $\mathcal{O}_P(C)$ & $\mathcal{O}_Q(C)$.

Prob 6.24 $\Rightarrow \exists f \in \mathfrak{m}_P(C)$ & $f^{-1} \in \mathcal{O}_Q(C)$

$\Rightarrow \mathrm{ord}(f) > 0$ & $\mathrm{ord}(f^{-1}) \geq 0$ ↯.

Existence : • We may assume

$C \hookrightarrow \mathbb{P}^n$, $C \cap U_i \neq \phi \ \forall i = 1, \cdots, n+1$.

(or, we may replace $\mathbb{P}^n$ with $\mathbb{P}^{n-1}$)

$\Rightarrow \Gamma_h(C) = k[X_1, \cdots, X_{n+1}]/I(C) = k[x_1, \cdots, x_{n+1}]$ $(x_i \neq 0)$

• $N := \max_{i,j} \operatorname{ord}(x_i/x_j)$

Assume $\operatorname{ord}(x_j/x_{n+1}) = N$, then

$\operatorname{ord}(x_i/x_{n+1}) = \operatorname{ord}(x_j/x_{n+1}) + \operatorname{ord}(x_i/x_j) = N - \operatorname{ord}(x_j/x_i) \geq 0$.

• $C_* :=$ affine curve corr. to $C \cap U_{n+1}$, then

$\Gamma(C_*) = k[x_1/x_{n+1}, \cdots, x_n/x_{n+1}] \subset R$

• $\mathfrak{m} =$ max. ideal of R $\quad J := \mathfrak{m} \cap \Gamma(C_*)$.

Prob 2.2 $\Rightarrow V(J) =$ closed subvar. W of C_*

$\Rightarrow W \subsetneq C_*$ $\left(\begin{array}{l} W = C_* \Rightarrow J = 0 \\ \Rightarrow \Gamma(C_*) \setminus \{0\} \subseteq R^\times \Rightarrow K \subseteq R \ \lightning \end{array} \right)$

$\Rightarrow W = \{P\}$ (pt !) (Prop 10. §6.5)

$\Rightarrow R$ dominates $\mathcal{O}_P(C_*) = \mathcal{O}_P(C)$

Cor 1. $f : \underset{\text{curve}}{C'} \dashrightarrow \underset{\text{proj. curve}}{C}$. Then

1) domain of f includes every simple pts of C'

2) $C' =$ nonsingular $\Rightarrow f =$ morphism

Pf: (1) ⇒ (2) : ✓

(1): $K = k(C)$, $L = k(C')$, $R := \mathcal{O}_P(C')$

we may assume f is dominating (or, by Pb 6.45 π is constant ⇒ ✓)

$\Rightarrow K \hookrightarrow L$

Thm 1 ⇒ ONTS: $K \not\subset R$.

Suppose NOT. Then $K \subset R \subset L$.

Prob 6.45 ⇒ L/K = f. alg. ext. $\overset{\text{Prob 1.50}}{\Rightarrow}$ R = field ↯. (DVR ≠ field!)

Cor 2. C = proj. curve, C' = nonsingular curve. Then

$$\{f : C' \to C \mid \text{dominant morphism}\} \overset{1:1}{\longleftrightarrow} \{\tilde{f} : k(C) \to k(C') \mid \text{homomorphism}\}$$

Cor 3. C, C' = nonsingular proj. curves.

$$C \cong C' \Leftrightarrow k(C) \cong k(C')$$

Cor 4. C = nonsingular proj. curve. $K = k(C)$.

$$\{P \in C\} \overset{1:1}{\longleftrightarrow} \{\text{DVR of } K\}$$

$$P \overset{\tau}{\longmapsto} \mathcal{O}_P(C).$$

Pf: $\mathcal{O}_P(C)$ = DVR ⇒ τ : well defined map

- τ injective: Thm 1
- τ surjective: $\forall R$ = DVR of K.

⇒ ∃! P s.t. R dominant $\mathcal{O}_P(C)$.

$$\mathcal{O}_P(C) \subseteq R \subseteq K$$

Prob 2.26 ⇒ $R = \mathcal{O}_P(C)$.

$K = k(C)$ nonsingular

$X := \{ R \subseteq K \mid \text{DVR}/k \}$

Top. on X : $\underset{\neq \emptyset}{U} \subset X$: open $\Leftrightarrow$ $X \setminus U$ = finite

$$\Rightarrow \quad \begin{array}{ccc} C & \longrightarrow & X \\ P & \longmapsto & \mathcal{O}_P(C) \end{array} \qquad \text{homeomorphism}$$

$$\Gamma(U, \mathcal{O}_C) = \bigcap_{P \in U} \mathcal{O}_P(C)$$

$\Rightarrow$ C is determined up to isomorphisms by K alone!

$\Rightarrow$ treat function fields avoid curve! (see Chevalley's "alg. functions of one variable")